§8 Direct Products, Central Products, Residual Products; Abelian grou Semi-simple groups, Wedderburn components of irreducible grou

(A) A group G is called the direct product of its subgroups $G_1, G_2, \cdots, G$ if (i) each element $\xi \in G$ is uniquely expressible in the form

$$\xi = \xi_1 \xi_2 \cdots \xi_n \qquad \text{with } \xi_i \in G_i \quad (i = 1, 2, \cdots, n)$$

and (ii) $[G_i, G_j] = 1$ for $i \neq j$.

If $\eta = \eta_1 \eta_2 \cdots \eta_n$ with $\eta_i \in G_i$, it follows from (ii) that $\xi\eta = \zeta$ has the expression $\zeta_1 \zeta_2 \cdots \zeta_n$ with $\zeta_i = \xi_i \eta_i$. Also $\xi^{-1} = \xi_1^{-1} \xi_2^{-1} \cdots \xi_n^{-1}$. Thus G is determined to within isomorphism by the direct factors $G_1, \cdots, G_n$.

Given n groups $G_1, \cdots, G_n$ not necessarily distinct, the set of all ordered multiplets $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ with $\xi_i \in G_i$ becomes a group if we define multiplication by the rule $\xi\eta = (\xi_1\eta_1, \cdots, \xi_n\eta_n)$. This implies that $\xi^{-1} = (\xi_1^{-1}, \cdots, \xi_n^{-1})$. This group is called the Cartesian product of $G_1, \cdots, G_n$ and is denoted by $G_1 \times G_2 \times \cdots \times G_n$. If G is the direct product of subgroups $G_1, \cdots, G_n$ we have a natural isomorphism of G onto the Cartesian product $G_1 \times \cdots \times G_n$.

In a direct product, the order of the direct factors is irrelevant. In a Cartesian product, ~~the~~ any permutation of the factors determines an ~~automorphism~~ isomorphism onto another Cartesian product.

If G is the direct product of the subgroups $(G_\alpha)_{\alpha \in \Lambda}$ and if Λ is expressed as the union of a number of disjoint subsets $\Lambda_1, \cdots, \Lambda_r$, then G is the direct product of the subgroups $G_1, \cdots, G_r$, where G_i is the product (also direct) of the G_α with $\alpha \in \Lambda_i$.

Lemma 8.1 Let G be generated by the subgroups $G_1, \cdots, G_n$. In order that G shall be the direct product of these subgroups it is necessary and sufficient that (i) each $G_i \triangleleft G$ and (ii) $G_i \cap G_1 G_2 \cdots G_{i-1} = 1$ for $i = 2, 3, \cdots, n$.

Proof: By (i), $G_i^* = G_1 G_2 \cdots G_{i-1}$ is a normal subgroup of G for each i. By (ii) and 6.9, it follows that $[G_i, G_i^*] = 1$ and so $[G_i, G_j] = 1$ for $i \neq j$. Then (ii) further ensures that each $\xi \in G$ is uniquely expressible in the form $\xi_1 \xi_2 \cdots \xi_n$ with $\xi_i \in G_i$.

(B) <u>Lemma 8.2</u> (i) A nilpotent group is the direct product of its Sylow subgroups.

(ii) If G is the direct product of subgroups $G_1, \cdots, G_n$ whose orders $m_1, \cdots, m_n$ are coprime in pairs, ~~then~~ and if H is any subgroup of G, then H is the direct product of the subgroups $H_i = H \cap G_i$ $(i = 1, 2, \cdots, n)$ and $\operatorname{Aut} G \cong \operatorname{Aut} G_1 \times \cdots \times \operatorname{Aut} G_n$.

(iii) If in (ii), the subgroups G_i are all cyclic, then G is cyclic.

(iv) If G is any group and ξ is any element of G of order $l = l_1 l_2 \cdots l_n$ where $(l_i, l_j) = 1$ for $i \neq j$, then $\xi = \xi_1 \xi_2 \cdots \xi_n$ with $[\xi_i, \xi_j] = 1$ for $i \neq j$ and with ξ_i of order l_i for each $i = 1, 2, \cdots, n$; moreover the elements $\xi_i \in G$ are uniquely determined by these conditions and are powers of ξ.

Proof: (i) Let $G_1, \cdots, G_n$ be the distinct Sylow subgroups $\neq 1$ of the nilpotent group G. Then $G_i \triangleleft G$ for each i, by 6·8 (vi). Also $|G| = \prod_i |G_i|$ and so $G = G_1 G_2 \cdots G_n$, and we have uniqueness in the expression $\xi = \xi_1 \cdots \xi_n$ of the elements $\xi \in G$ with $\xi_i \in G_i$. Also $[G_i, G_j] = 1$ for $i \neq j$, by 6·9. The result now follows.

(ii) If ϖ_i is the set of primes dividing m_i, then G_i is a normal S_{ϖ_i}-subgroup of G. We have $H_i \triangleleft H$ and $H_i \leq G_i$ so $[H_i, H_j] = 1$ for $i \neq j$, since then $[G_i, G_j] = 1$. $H/H_i \cong G_i H / G_i$ and so $|H : H_i|$ is a ϖ_i'-number since $|G_i H : G_i|$ divides $|G : G_i| = m_1 m_2 \cdots m_{i-1} m_{i+1} \cdots m_n$. Hence H_i is a normal S_{ϖ_i}-subgroup of H. Since $|H|$ divides $|G| = m_1 m_2 \cdots m_n$ it follows that $H = H_1 H_2 \cdots H_n$ and uniqueness for the expression of the elements of H in the form $\xi = \xi_1 \cdots \xi_n$ with $\xi_i \in H_i$ follows from the corresponding result for G. Hence H is the direct product of $H_1, \cdots, H_n$.

Let $\alpha \in A = \operatorname{Aut} G$. Since G_i char G by 5·8, α leaves each G_i invariant and induces in G_i an automorphism α_i. α is uniquely determined by $\alpha_1, \alpha_2, \cdots, \alpha_n$ since $G = G_1 G_2 \cdots G_n$. Given $\xi = \xi_1 \cdots \xi_n \in G$ with $\xi_i \in G_i$ and given $\alpha_i \in A_i = \operatorname{Aut} G_i$, the mapping $\xi \to \xi^\alpha$ of G

defined by $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$ is an automorphism of G which induces α_i on G_i. If α and $\beta \in A$, then $(\alpha\beta)_i = \alpha_i \cdot \beta_i$ for each i. Hence $A \cong A_1 \times \cdots \times A_n$, as stated.

(iii) If H is cyclic of order $m_1 m_2 \cdots m_n$ with $(m_i, m_j) = 1$ for $i \neq j$, then H has a subgroup H_i of order m_i and H_i is cyclic by 2.5. By (B) H is the direct product of $H_1, \ldots, H_n$. Hence $H \cong H_1 \times \cdots \times H_n$ and any direct product of cyclic subgroups of orders $m_1, \ldots, m_n$ is isomorphic with H, hence cyclic.

(iv). Take $H = \{\xi\}$ with $m_i = l_i$, $i = 1, 2, \ldots, n$. Then $\xi = \xi_1 \cdots \xi_n$ with $\xi_i \in H_i$, $|H_i| = l_i$. The order of ξ_i is precisely l_i since otherwise the order of ξ would be less than $l = l_1 l_2 \cdots l_n$. Each ξ_i is a power of H, hence $[\xi_i, \xi_j] = 1$ for all i, j. Conversely, given elements $\xi_1, \ldots, \xi_n$ in G with these properties, we have $\xi^r = \xi_1^r \cdots \xi_n^r$ and so $\xi^{l/l_i} = \xi_i^{l/l_i}$, which has order l_i since otherwise ξ would be of order less than l. Hence ξ_i is a power of ξ_i^{l/l_i} and therefore $\xi_i \in H$ for each i. Thus $\xi_1, \ldots, \xi_n$ are uniquely determined by ξ.

(C) 8·2 (ii) indicates ~~shows~~ that the properties of direct products of groups whose orders are coprime are easily reducible to a study of the structure of the direct factors. ~~[illegible]~~ In particular, the theory of nilpotent groups reduces trivially to the theory of p-groups. When the direct factors are not of coprime orders, however, more difficult questions arise. For example, in this case, the direct factors need not be characteristic subgroups.

Consider the subgroups H of the direct product $G = G_1 G_2$ of two groups G_1, G_2. Let $H_i = H \cap G_i$ $(i = 1, 2)$ and let $\overline{H}_1 = G_1 \cap HG_2$, $\overline{H}_2 = G_2 \cap HG_1$. Let $\xi = \eta \gamma_2 \in \overline{H}_1$ where $\eta \in H$, $\gamma_2 \in G_2$ and let $\alpha \in H_1$. Then $\alpha^\eta \in H_1 = H \cap G_1$ since $\alpha \in G_1 \lhd G$, and $\alpha^\xi = \alpha^\eta$ since $[G_1, \gamma_2] = 1$. Hence $H_1 \lhd \overline{H}_1$ and similarly $H_2 \lhd \overline{H}_2$. If $\xi = \xi_1 \xi_2 \in G$ with $\xi_i \in G_i$, then the mapping $\xi \to \xi_i$ is homomorphic. Call this mapping θ_i $(i = 1, 2)$. We have $\theta_i(H) = \overline{H}_i$ and the restriction of

θ_2 to H has kernel H_2, so that $\overline{H}_1 \cong H/H_2$; and similarly $\overline{H}_2 \cong H/H_1$. Suppose $\xi = \xi_1\xi_2$ and $\eta = \eta_1\eta_2$ lie in H where ξ_i, η_i are in G_i and hence in $\overline{H}_i$. Then $\xi\eta^{-1} = (\xi_1\eta_1^{-1})(\xi_2\eta_2^{-1}) \in H$. Hence $\xi_1\eta_1^{-1} \in H_1$ implies $\xi_2\eta_2^{-1} \in H_2$ and conversely. Hence the correspondence $H_1\xi_1 \to H_2\xi_2$ is one-to-one and is an isomorphism mapping $\overline{H}_1/H_1$ onto $\overline{H}_2/H_2$.

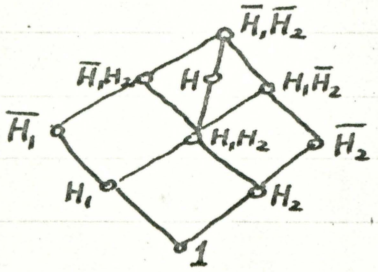
Conversely, let $\overline{H}_1/H_1$ and $\overline{H}_2/H_2$ be any two isomorphic sections of G_1 and G_2 respectively and let θ be any isomorphism of the first onto the second. Let H be the set of all $\xi = \xi_1\xi_2 \in G$ such that $\xi_i \in \overline{H}_i$ and $(H_1\xi_1)^\theta = H_2\xi_2$. If $\eta = \eta_1\eta_2$ also lies in H, then $\xi_i\eta_i \in \overline{H}_i$ and $(H_1\xi_1\eta_1)^\theta = (H_1\xi_1)^\theta(H_1\eta_1)^\theta = (H_2\xi_2)(H_2\eta_2) = H_2\xi_2\eta_2$ and so $\xi\eta \in H$. Thus H is a subgroup of G. Thus we can state

<u>Theorem 8.3</u> Let G be the direct product of the subgroups G_1 and G_2. Then there is a one-to-one correspondence between the subgroups H of G and the set of all isomorphisms θ of a section $\overline{H}_1/H_1$ of G_1 onto a section $\overline{H}_2/H_2$ of G_2. In this correspondence, $H_i = H \cap G_i$ and $\overline{H}_1 = G_1 \cap HG_2$, $\overline{H}_2 = G_2 \cap HG_1$. An element $\xi = \xi_1\xi_2$ of G with $\xi_i \in G_i$ lies in H if and only if $\xi_i \in \overline{H}_i$ $(i=1,2)$ and $(H_1\xi_1)^\theta = H_2\xi_2$. (Goursat)

(D). It is important to consider the case in which G admits a group of operators Γ leaving the direct factors G_1 and G_2 of G invariant. We wish to know in this case which subgroups H of G are Γ-invariant. If H is Γ-invariant, then so are the subgroups H_i and $\overline{H}_i$. Hence any $\alpha \in \Gamma$ induces an automorphism in each of the two ~~fact~~ sections $\overline{H}_1/H_1$ and $\overline{H}_2/H_2$. If $\xi_1\xi_2 = \xi \in H$ with $\xi_i \in \overline{H}_i$, we then have also $\xi^\alpha = \xi_1^\alpha\xi_2^\alpha \in H$ and hence $(H_1\xi_1^\alpha)^\theta = H_2\xi_2^\alpha = (H_1\xi_1)^{\theta\alpha}$. In other words, the isomorphism θ of 8.3 must commute with every $\alpha \in \Gamma$. Conversely, if this condition is fulfilled in addition to the Γ-invariance of the $H_i, \overline{H}_i$, then H is clearly Γ-invariant. This gives, as an essential supplement to 8.3,

<u>Theorem 8.4</u> If in 8.3 the group G admits a group of operators Γ leaving G_1 and G_2 invariant, then the subgroup H of G is Γ-invariant if and only if the subgroups $\bar{H}_i$, H_i $(i=1,2)$ are also Γ-invariant and in addition $(H_1\xi_1)^{\alpha\theta} = (H_1\xi_1)^{\theta\alpha}$ for all $\alpha \in \Gamma$ and $\xi_1 \in \bar{H}_1$. In particular, $H \lhd G$ if and only if $\bar{H}_i/H_i$ is a central factor of G_i for $i=1,2$.

In the case $H \lhd G$, we have $\Gamma = G$ and so the H_i and $\bar{H}_i$ must be normal in G or, what is equivalent, normal in G_i $(i=1,2)$. In addition, taking $\alpha \in G_1$ we find that $(H_1\xi_1^{\alpha})^{\theta} = (H_1\xi_1)^{\theta\alpha} = (H_1\xi_1)^{\theta}$ since then $[G_2, \alpha] = 1$. Hence $H_1\xi_1^{\alpha} = H_1\xi_1$ for all $\alpha \in G_1$ and $\xi_1 \in \bar{H}_1$ so that $\bar{H}_1/H_1$ is a central factor of G_1. Similarly, taking $\beta \in G_2$, we have $(H_1\xi_1)^{\theta\beta} = (H_1\xi_1)^{\beta\theta} = (H_1\xi_1)^{\theta} = H_2\xi_2$ say, so that $H_2\xi_2^{\beta} = H_2\xi_2$ for all $\beta \in G_2$ and $\xi_2 \in \bar{H}_2$. Thus $\bar{H}_2/H_2$ is a central factor of G_2. Conversely, these conditions are sufficient to ensure that $(H_1\xi_1)^{\gamma\theta} = (H_1\xi_1)^{\theta\gamma}$ for all $\gamma \in G$, and so they imply $H \lhd G$.



Note that $H \lhd G$ implies $H_1H_2 \lhd G$ and G/H_1H_2 is the direct product of G_1H_2/H_1H_2 with ~~G~~ H_1G_2/H_1H_2. It is therefore isomorphic with the ~~direct~~ Cartesian product $G_1/H_1 \times G_2/H_2$. In this Cartesian product $\bar{H}_1/H_1 \times \bar{H}_2/H_2$ is a subgroup of the centre $\zeta(G_1/H_1) \times \zeta(G_2/H_2)$. Thus the step from G to G/H may be thought to take place in two steps. First we form the direct product $\bar{G}_1\bar{G}_2$ where $\bar{G}_i \cong G_i/H_i$. Then $G/H \cong \bar{G}_1\bar{G}_2/K$, where K is a subgroup of the centre of $\bar{G}_1\bar{G}_2$ consisting of all elements $\bar{\xi}_1\bar{\xi}_2$ with $\bar{\xi}_i \in K_i \leq \zeta\bar{G}_i$ with $\bar{\xi}_2 = \bar{\xi}_1^{\theta}$, where θ is an isomorphism of K_1 onto K_2, K_1 and K_2 being necessarily isomorphic subgroups of $\zeta\bar{G}_1$, $\zeta\bar{G}_2$ respectively. The group $\bar{G}_1\bar{G}_2/K \cong$ is thus obtained from the direct product $\bar{G}_1\bar{G}_2$ by "identifying" correspondin

$\bar{\xi}_1 = (\bar{\xi}_2)^{-1}$ is an isomorphism $\bar{\xi}_1 \to (\bar{\xi}_2)^{-1}$ of a subgroup K_1 of $\mathfrak{z}\bar{G}_1$ onto a subgroup K_2 of $\mathfrak{z}\bar{G}_2$. A group constructed in this way is often called a <u>central product</u> of the two groups $\bar{G}_1$ and $\bar{G}_2$.

The last part of 8·4 may now be stated as follows: any homomorphic image G/H of the direct G of two groups G_1 and G_2 is isomorphic with some central product of homomorphic images $\bar{G}_1 = G_1/H_1$ and $\bar{G}_2 = G_2/H_2$ of G_1 and G_2.

(E). Let H be a subgroup of the Cartesian product $G = G_1 \times G_2 \times \cdots \times G_n$. If for each $i = 1, 2, \ldots, n$ and each $\xi_i \in G_i$, there is an element of H whose i-th coordinate is precisely ξ_i, then H is called a <u>residual product</u> of $G_1, G_2, \ldots, G_n$; or sometimes though less appropriately a <u>subdirect produ</u>[ct] of the G_i. If $\xi = (\xi_1, \ldots, \xi_n) \in G$, the mapping $\theta_i : \xi \to \xi_i$ is a homomorphism of G onto G_i. H is a residual product of the G_i if and only if $H^{\theta_i} = G_i$ for each i.

<u>Lemma 8·5</u> Let $K_i \lhd G$ $(i = 1, 2, \ldots, n)$ and let $K = \bigcap_{i=1}^{n} K_i$. Then G/K is isomorphic with a residual product of the groups G/K_i $(i = 1, 2, \ldots, n)$.

The mapping

$$K\xi \to (K_1\xi, \ldots, K_n\xi) \qquad (\xi \in G)$$

is the isomorphism in question. For this mapping is homomorphic owing to $K_i \lhd G$ for each i. If $K\xi$ and $K\eta$ have the same image, then $K_i\xi = K_i\eta$ for all i and so $\xi\eta^{-1} \in K = \bigcap_i K_i$, whence $K\xi = K\eta$. Thus the mapping is an isomorphism. The image group is obviously a residual product of the G/K_i.

(F) **Lemma 8.6** Let G be an Abelian p-group and let ξ be any element of G whose order is as large as possible. Then G is the direct product of $X = \{\xi\}$ and Y, where Y is any subgroup of G which is maximal with respect to the condition $X \cap Y = 1$.

Proof: We have only to show that $XY = G$ since G is Abelian. Let $\overline{G} = G/Y$, $\overline{X} = XY/Y \cong X$. By the maximal property of Y, every subgroup $\overline{H} \neq 1$ in $\overline{G}$ contains elements $\neq 1$ of the cyclic subgroup $\overline{X} = \{\overline{\xi}\}$. Let $\overline{\eta}$ be any element of $\overline{G}$ and let $\overline{\eta}^{p^r}$ be the first positive power of $\overline{\eta}$ to lie in $\overline{X}$, i.e. $\{\overline{\eta}^{p^r}\} = \{\overline{\eta}\} \cap \overline{X}$. Since $|\overline{X}| = |X|$, the order of $\overline{\eta}$ cannot exceed the order of $\overline{\xi}$, by our choice of ξ. Hence $\{\overline{\eta}^{p^r}\} = \{\overline{\xi}^{p^s}\}$ for some $s \geq r$, and so for a suitable integer m, the element $\overline{\eta}\,\overline{\xi}^{m}$ has order p^r and $\overline{H} = \{\overline{\eta}\,\overline{\xi}^m\}$ then has intersection 1 with $\overline{X}$. Hence $\overline{\eta}\,\overline{\xi}^{m} = 1$ and so $\overline{\eta} \in \overline{X}$. Thus $\overline{G} = \overline{X}$ and $G = XY$ as required.

In an Abelian p-group G, the elements ξ such that $\xi^{p^m} = 1$ form a characteristic subgroup $\Omega_m G$ and the elements η of the form $\eta = \xi^{p^m}$ for some $\xi \in G$ form another characteristic subgroup $\mho_m G$.

An immediate corollary of 8.6 is

Theorem 8.7 (i) Every Abelian p-group G is the direct product of a certain number of cyclic subgroups $X_i = \{\xi_i\}$ $(i = 1, 2, \ldots, r)$. If $|X_i| = p^{\lambda_i}$, we can arrange that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$. The partition λ whose parts are $\lambda_1, \lambda_2, \ldots, \lambda_r$ is called the <u>type</u> of G and the numbers λ_i are the <u>invariants</u> of G. The ordered set $\xi_1, \ldots, \xi_r$ is called a <u>basis</u> of G.

(ii) Two Abelian p-groups are isomorphic if and only if they have the same type.

Proof: (i) follows from 8.6 by induction on $|G|$, since we can then assume $X_1 = X$ and that Y is the direct product of suitable cyclic subgroups $X_2, \ldots$.

(ii) Every element of G is uniquely expressible in the form

$$\xi = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_r^{\alpha_r} \quad \text{with } 0 \leq \alpha_i < p^{\lambda_i}.$$

$\xi \in \Omega_m G$ if and only if $\alpha_i p^m \equiv 0 \mod p^{\lambda_i}$ for each i, which means p^m choices for α_i if $\lambda_i \geqslant m$ and p^{λ_i} choices if $\lambda_i \leq m$. Hence $|\Omega_m G| = p^{\rho_1 + \rho_2 + \cdots + \rho_m}$, where ρ_m is the number of values of i for which $\lambda_i \geqslant m$. We have $\rho_1 \geqslant \rho_2 \geqslant \cdots \geqslant \rho_l > \rho_{l+1} = 0$, where $l = \lambda_1$, $r = \rho_1$. The partition ρ whose parts are $\rho_1, \ldots, \rho_l$ is the <u>conjugate partition</u> to λ and the relation between ρ and λ is symmetrical: $\rho = \lambda'$, $\rho' = \lambda$. Now G determines ρ uniquely because $|\Omega_m G : \Omega_{m-1} G| = p^{\rho_m}$ for $m = 1, 2, \ldots$. Hence G determines λ uniquely.

Note that the mapping $\xi \to \xi^{p^m}$ $(\xi \in G)$ is a homomorphism of G onto $\mho_m G$ with kernel $\Omega_m G$. Hence

$$G/\Omega_m G \cong \mho_m G \quad \text{and} \quad |\mho_{m-1} G : \mho_m G| = p^{\rho_m} = |\Omega_m G : \Omega_{m-1} G|.$$

<u>Corollary 8.71</u> Every finite Abelian group G is the direct product of cyclic subgroups of orders $h_1, h_2, \ldots, h_r$ such that $h_r > 1$ and h_{i+1} divides h_i for each $i = 1, \ldots, r-1$. The numbers $h_1, \ldots, h_r$ are uniquely determined by G subject to these conditions. They are called the <u>elementary divisors</u> of G.

Note that if G is an Abelian p-group of type λ, then the elementary divisors of G are the numbers $p^{\lambda_1}, p^{\lambda_2}, \ldots, p^{\lambda_r}$ where $\lambda_1, \ldots, \lambda_r$ are the parts of λ.

Proof: By 8.2 (i), G is the direct product of its Sylow subgroups $\neq$ say $G_1, \ldots, G_s$ where $|G_j| = p_j^{n_j}$. Let the invariants of G_j be $\lambda_{j1} \geqslant \lambda_{j2} \geqslant \cdots \geqslant \lambda_{jr_j} > 0$ and let $r = \max r_j$ $(j = 1, 2, \ldots, s)$. Let $\xi_{j1}, \ldots, \xi_{jr_j}$ be a basis of G_j and define $\xi_{jk} = 1$ if $r_j < k \leq r$. Let $\xi_i = \xi_{1i} \xi_{2i} \cdots \xi_{si}$ $(i = 1, 2, \ldots, r)$. Then $X_i = \{\xi_i\}$ is of order $h_i = p_1^{\lambda_{1i}} p_2^{\lambda_{2i}} \cdots p_s^{\lambda_{si}}$ where $\lambda_{ji} = 0$ if $r_j < i \leq r$. Also X_i is the direct product of the cyclic groups $\{\xi_{1i}\}, \{\xi_{2i}\}, \ldots, \{\xi_{si}\}$. Hence G is the direct product of $X_1, \ldots, X_r$ and by construction the numbers $h_1, \ldots, h_r$ satisfy the conditions of 8.71. Conversely, these conditions ensure that $\lambda_{j1} \geqslant \lambda_{j2} \geqslant \cdots \geqslant \lambda_{jr} \geqslant 0$ for each $j = 1, 2, \ldots, s$ and therefore imply that the Sylow p_j-subgroup G_j of G has type $\lambda^{(j)}$ ~~with~~ where

$\lambda^{(j)}$ is the partition whose parts are those λ_{ji} which are positive. Hence the elementary divisors $h_1, \ldots, h_r$ are uniquely determined by G.

(G). An Abelian p-group G of type (1^n) is called <u>elementary</u>. An Abelian p-group G is elementary if and only if it has no elements of order p^2.

If $\xi_1, \ldots, \xi_r$ is a basis of the Abelian p-group G and if $\alpha \in \operatorname{Aut} G$ then $\xi_1^\alpha, \ldots, \xi_r^\alpha$ is also a basis of G. $\xi_i^\alpha = \xi_i$ for all $i = 1, 2, \ldots, r$ implies that α is the identity on G. If $\eta_1, \ldots, \eta_r$ is an arbitrary basis of G, then the mapping $\xi_i \to \eta_i$ $(i = 1, 2, \ldots, r)$ determines uniquely an automorphism β of G viz.

$$\xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_r^{\alpha_r} \longrightarrow \eta_1^{\alpha_1} \eta_2^{\alpha_2} \cdots \eta_r^{\alpha_r} \qquad (0 \leq \alpha_i < p^{\lambda_i})$$

where λ is the type of G. Hence the number $|\operatorname{Aut} G|$ of automorphisms of G is equal to the number of bases of G.

When G is elementary, $\eta_1, \eta_2, \ldots, \eta_n$ is a basis of G if and only if $Y_s = \{\eta_1, \eta_2, \ldots, \eta_s\}$ is of order p^s for each $i = 1, 2, \ldots, n$. Given Y_{s-1}, this leaves precisely $p^n - p^{s-1}$ choices for η_s. Hence $|\operatorname{Aut} G|$ is equal to $(p^n - 1)(p^n - p) \cdots (p^n - p^{n-1})$.

Suppose next that G has n invariants all equal to m, so that $|G| = p^{mn}$ and $G/\mho_1 G$ is elementary of order p^n. If $\xi_1, \ldots, \xi_n$ is a basis of G, then $H\xi_1, \ldots, H\xi_n$ generates G/H where H is any subgroup of G. Taking $H = \mho_1 G$, this implies that $H\xi_1, \ldots, H\xi_n$ is then a basis of G/H. Conversely, if this is so, then each ξ_i has order p^m and the group $X = \{\xi_1, \ldots, \xi_n\}$ coincides with G. For if $X < G$, then there is a subgroup Y of index p in G such that $X \leq Y$ and we have $H = \mho_1 G \leq Y$, whence $HX < G$ and $H\xi_1, \ldots, H\xi_n$ could not be a basis of G/H. Since $|H| = p^{n(m-1)}$, we thus have $|\operatorname{Aut} G| = p^{n^2(m-1)}(p^n - 1)(p^n - p) \cdots (p^n - p^{n-1})$ in this case. Note that the automorphisms α of G which transform the characteristic quotient group $G/\mho_1 G$ identically form a normal subgroup A_1 of $A = \operatorname{Aut} G$, that $|A_1| = p^{n^2(m-1)}$ and that A/A_1 is isomorphic with

Aut $G/\mho_1 G$.

Now let G be any Abelian p-group, let $\xi_1, \dots, \xi_r$ and $\eta_1, \dots, \eta_r$ be any two bases of G and suppose that G has exactly n invariants equal to m. Let $\xi_{a+1}, \xi_{a+2}, \dots, \xi_{a+n}$ be those ξ's which have order p^m and let $X = \{\xi_1, \xi_2, \dots, \xi_a, \xi_{a+n+1}, \dots, \xi_r\}$ be the subgroup generated by the remaining ξ's. Then no element of order p in X can belong to the set $\mho_{m-1}G - \mho_m G$. On the other hand, every element of order p in $Y = \{\eta_{a+1}, \eta_{a+2}, \dots, \eta_{a+n}\}$ lies in $\mho_{m-1}G - \mho_m G$. Hence $X \cap Y = 1$ and so G is the direct product of the two subgroups X and Y. It follows that

$$\xi_1, \xi_2, \dots, \xi_a, \eta_{a+1}, \eta_{a+2}, \dots, \eta_{a+n}, \xi_{a+n+1}, \dots, \xi_r$$

is a basis of G. In other words, we can exchange the elements of given order p^m in the ξ-basis for the corresponding elements of the η-basis and obtain a new basis. This is called the <u>exchange property</u> of the bases of an Abelian p-group.

Now the number of different subgroups $Y, Y_1, Y_2, \dots$ of G such that G is the direct product of X with Y_i, i.e. the number of subgroups Y_i such that $X \cap Y_i = 1$ and $XY_i = G$, is equal to the number of distinct homomorphisms of Y into X. This is a particular case of 8.3. Hence the number of different ordered sets $\eta_{a+1}, \dots, \eta_{a+n}$ of elements of order p^m which can occur in some basis or other of G is equal to

$$a_m = |\mathrm{Hom}\,(Y, X)|\,.\,|\mathrm{Aut}\,Y|. \qquad (1)$$

And by the exchange property, $|\mathrm{Aut}\,G| = \prod_{m=1}^{\infty} a_m$, where $a_k = 1$ if G has no invariants equal to k.

<u>Lemma 8.8</u> Let X and Y be Abelian groups, let α and β be homomorphisms of Y into X and define the sum $\alpha+\beta$ to be the mapping $\eta \to \eta^{\alpha+\beta} = \eta^\alpha \eta^\beta$ $(\eta \in Y)$. Then $\alpha+\beta \in H = \mathrm{Hom}\,(Y, X)$ and with this addition, H becomes an additive (Abelian) group. If X and Y are Abelian p-groups of types λ and μ respectively, then H is an Abelian p-group of type $\lambda \otimes \mu$, where the parts of $\lambda \otimes \mu$ are the numbers $\min(\lambda_i, \mu_j)$, $(i = 1, 2, \dots, r;\ j = 1, 2, \dots, s)$, and where

λ and μ have r and s parts respectively.

Note that if $\nu = \lambda \otimes \mu$, then the parts of the conjugate ν' of ν are the numbers $\lambda'_1\mu'_1, \lambda'_2\mu'_2, \cdots, \lambda'_t\mu'_t$, where $t = \min(\lambda_1, \mu_1)$ and λ', μ' are the partitions conjugate to λ, μ respectively.

The verification that H is an additive Abelian group is immediate. The zero element of H maps Y into the unit subgroup of X. In the p-group case, if $\xi_1, \cdots, \xi_r$ and $\eta_1, \cdots, \eta_s$ are bases of X and Y respectively, then the elements α_{ij} $(i = 1, \cdots, r\,;\ j = 1, \cdots, s)$ of H defined by

$$\alpha_{ij} : \eta_j \rightarrow \xi_i^{p^{\max(0, \lambda_i - \mu_j)}} \quad ; \ \eta_k \rightarrow 1 \ \ (k \neq j)$$

form a basis of H. The order of α_{ij} is precisely $p^{\min(\lambda_i, \mu_j)}$. The number of pairs (i, j) such that $\min(\lambda_i, \mu_j) \geqslant m$ is equal to $\lambda'_m\mu'_m$.

Note that $\mathrm{Hom}(Y, X)$ and $\mathrm{Hom}(X, Y)$ have the same type. An Abelian group isomorphic with $H = \mathrm{Hom}(Y, X)$ is often called the <u>tensor produ</u> of X and Y.

We now use 8·8 and equation (1) to calculate $|\mathrm{Aut}\, G|$ for an Abelian p-group G of arbitrary type $\lambda = (1^{r_1} 2^{r_2} 3^{r_3} \cdots)$. Here Y is of type (m^{r_m}) with r_m invariants equal to m. Define the polynomial f_n by the equation

$$f_n(x) = (1-x)(1-x^2) \cdots (1-x^n) \qquad (n = 1, 2, \cdots)$$

and understand $f_0(x) = 1$. Then we have seen that $|\mathrm{Aut}\, Y| = p^{m r_m^2} f_{r_m}(\tfrac{1}{p}$ The conjugate of the partition (m^{r_m}) is the partition $(r_m, r_m, \cdots, r_m)$ with m parts all equal to r_m. The conjugate of the type of X has parts $\lambda'_1 - r_m, \lambda'_2 - r_m, \cdots, \lambda'_m - r_m, \lambda'_{m+1}, \cdots$. Hence $H = \mathrm{Hom}(Y, X)$ has type ν where ν' has the parts $r_m(\lambda'_1 - r_m), r_m(\lambda'_2 - r_m), \cdots, r_m(\lambda'_m - r_m)$. Now $r_m = \lambda'_m - \lambda'_{m+1}$ and $|H| = p^{-m r_m^2 + \sum_{i=1}^{m} r_m \lambda'_i}$. Hence $|\mathrm{Aut}\, G| = p^g \prod_{m=1}^{\infty} f_{r_m}(\tfrac{1}{p})$ where $g = \sum_{\substack{i \leqq m \\ i, m = 1, 2, 3, \cdots}} r_m \lambda'_i = \sum_{i=1}^{\infty} (\lambda'_i)^2$. Thus we obtain

<u>Corollary 8·81</u> Let G be an Abelian p-group of type λ, let g be the sum of the squares of the parts of the partition conjugate to λ and let $f_n(x) = (1-x)(1-x^2) \cdots (1-x^n)$, with $f_0(x) = 1$. Then

$|\mathrm{Aut}\, G| = p^g \prod_{m=1}^{\infty} f_{r_m}(\frac{1}{p})$, where $\lambda = (1^{r_1} 2^{r_2} \ldots)$ and so the numbers r_m are the multiplicities of the different parts of λ.

(H) A group is called <u>semisimple</u> if it is the direct product of one or more simple subgroups each of composite order, or if it is 1.

<u>Lemma 8.91</u> (i) Let H be a semisimple group. Then every subnormal subgroup of H is normal and is a direct factor of H and is itself semisimple.

(ii) Let H and K be normal semisimple subgroups of G, let $M = H \cap K$ and let $L = C_{HK}(M)$. Then HK is semisimple and is the direct product of L and M.

Proof: (i) Let the direct factors of H be $H_1, H_2, \ldots, H_n$, each H_i being simple of composite order, and let $\xi = \xi_1 \xi_2 \cdots \xi_n$ with $\xi_i \in H_i$. Then if $\eta_i \in H_i$, we have $[\xi, \eta_i] = [\xi_i, \eta_i]$. If $\xi_i \neq 1$, it follows that $X = \{\xi^H\}$ contains H_i, since H_i is simple but not Abelian. Hence every normal subgroup M of H is the product of certain of the simple direct factors H_i. Thus M is also semisimple. It then follows that every subnormal subgroup of H is a direct factor of H. Clearly H has exactly 2^n distinct direct factors.

(ii) Since $K \triangleleft G$, we have $M \triangleleft H$ and so $H = ML_1$ is the direct product of M with a subgroup L_1 and both M and L_1 are semisimple. Similarly K is the direct product of M with a semisimple subgroup L_2. We have $L_1 \leq L$ and $L_2 \leq L$ and so $[L_1, L_2] \leq [H, K] \cap L$. But $[H, K] \leq M$, by the normality of H and K; while $M \cap L = 1$ since $\mathfrak{z}M = 1$. Hence $[L_1, L_2] = 1$. But $L_1 \cap L_2 \leq L \cap M = 1$ and so the product $L_1 L_2$ is direct. Since $|HK| = |M| \cdot |L_1| \cdot |L_2|$, we have $HK = ML_1L_2$; and $L \cap M = 1$ ensures that $L = L_1 L_2$ is the direct product of the semisimple groups L_1 and L_2, hence itself semisimple; while HK is the direct product of L and M, since L and M are normal in G.

Let $\mathfrak{X}$ be any class of groups, such that (i) all groups of order 1 belong to $\mathfrak{X}$; and (ii) if $G \in \mathfrak{X}$ and $G \cong G_1$, then $G_1 \in \mathfrak{X}$. We shall always understand the expression "class of groups" to imply (i) and (ii).

<u>Lemma 8.92</u> Suppose that in ~~any~~ every group G the product of two normal $\mathfrak{X}$-subgroups always belongs to $\mathfrak{X}$; and that every normal subgroup of an $\mathfrak{X}$-group is an $\mathfrak{X}$-group. Then every group G has a uniquely determined maximal normal $\mathfrak{X}$-subgroup $\mathfrak{X}G$. If H sbn G, then $\mathfrak{X}H = H \cap \mathfrak{X}G$. In particular, $\mathfrak{X}G$ contains every ~~normal~~ subnormal $\mathfrak{X}$-subgroup of G.

The argument is the same as for 7.61, which is the special case $\mathfrak{X} = \mathfrak{N}$ the class of all nilpotent groups. 8.91 shows that $\mathfrak{X} = \mathfrak{S}$ the class of all semisimple groups is another admissible choice. A third choice would be $\mathfrak{X} = \mathfrak{O}_{\varpi}$ the class of all ϖ-groups. This is admissible by 5.5 or 3.4, which shows that products of normal ϖ-subgroups are ϖ-groups.

The subgroup $\mathfrak{X}G$ is characteristic in G and may be called the <u>$\mathfrak{X}$-radical</u> of G. Here "radical" is equivalent to "uniquely determinal maximal normal subgroup" of the appropriate class.

(I). A semisimple group will be called <u>isotypic</u> if its simple direct factors are all isomorphic.

<u>Lemma 8.93</u> A characteristically-simple group G is either (i) an elementary Abelian p-group for some prime p or else (ii) an isotypic semisimple group.

Proof: let G_1 be a minimal normal subgroup of G and let H be the direct product of $G_1, G_2, \ldots, G_n$ where each $G_i = G_1^{\alpha_i}$ for some $\alpha_i \in A = \mathrm{Aut}\, G$. Choose H as large as possible and let $\beta \in A$. If $H^\beta \neq H$, then $G_i^\beta \not\leq H$ for some i. But $H \lhd G$ and G_i^β is a minimal normal subgroup of G. Hence $H \cap G_i^\beta = 1$ and the product HG_i^β is direct by 6.9, contrary to the definition of H. It follows that $H^\beta = H$ for all $\beta \in A$ and so H char G. Since G is characteristically simple and $H \geqslant G_1 \neq 1$, it follows that $H = G$. Each $G_i \cong G_1$ and every ~~...~~ normal subgroup of G_1 is normal in G. Hence G_1 is simple. If $|G_1| = p$ is prime, then G is elementary. Otherwise G is isotypic and semisimple.

(J) Let V be an elementary p-group which admits G as a group of operators. We use the additive notation for V. V is called <u>G-irreducible</u> if $V \neq 0$ and if V and 0 are the only G-invariant subgroups of V. In any case if $V \neq 0$, a minimal G-invariant subgroup $X \neq 0$ in V is necessarily G-irreducible.

<u>Theorem 8.9</u>. Let V be G-irreducible and let $H \triangleleft G$. Then $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ is the direct sum of a certain number $r \geqslant 1$ of H-invariant subgroups V_i with the following properties:

(i) Each V_i is the direct sum of a certain number $\rho \geqslant 1$ of H-irreducible subgroups ~~...~~ $X_{i1}, X_{i2}, \ldots, X_{i\rho}$, where ρ is independent of i.

(ii) X_{is} and X_{jt} are H-isomorphic if and only if $i = j$.

(iii) If Y is any H-invariant subgroup of V, then $Y = Y_1 \oplus \cdots \oplus Y_r$ where $Y_i = Y \cap V_i$. In particular, if Y is H-irreducible, then Y is H-isomorphic with X_{i1} for some i and in that case $Y \leqslant V_i$.

(iv) If $\xi \in G$, then the mapping $V_i \to V_i\xi$ $(i = 1, 2, \ldots, r)$ is a permutation of $V_1, \ldots, V_r$ and G is represented transitively by these permutations.

Proof: Let $W = W_1 \oplus \cdots \oplus W_n$ be a direct sum of H-irreducible subgroups W_i of V, and choose W as large as possible. Let $\xi \in G$. Every subgroup of $W_i\xi$ has the form $Z\xi$ where Z is a subgroup of W_i. If $\eta \in H$, $u \in Z$ then $u\xi\eta = u(\xi\eta\xi^{-1})\xi$ and, since $H \triangleleft G$, $Z\xi$ is H-invariant only if Z is H-invariant. Hence $W_i\xi$ is H-irreducible. If $W\xi \neq W$, then $W_i\xi \nleqslant W$ for some i and hence $W \cap W_i\xi = 0$ and the sum $W + W_i\xi$ is direct, contrary to the choice of W. It follows that $W\xi = W$ for all $\xi \in G$. Since V is G-irreducible, we must therefore have $W = V$.

We now relabel the subgroups W_j as X_{is} so that (ii) is satisfied and define V_i as the direct sum of the X_{is} $(s = 1, 2, \dots)$, so that $V = V_1 \oplus V_2 \oplus \dots \oplus V_r$ if there are r classes of H-isomorphic H-irreducible subgroups among the W_j. To prove (iii), suppose $W_j \not\leq Y$. Then $Y \cap W_j = 0$ since W_j is H-irreducible and Y is H-invariant; and so the sum $Y + W_j$ is direct. It follows that $V = Y \oplus W_{j_1} \oplus \dots \oplus W_{j_t}$ for some $t \geqslant 0$ and suitable $j_1, \dots, j_t$. Hence Y is H-isomorphic with $V / \sum_{\alpha=1}^{t} W_{j_\alpha}$ and so with $W_{i_1} \oplus \dots \oplus W_{i_s}$, where $i_1, \dots, i_s$ is the complementary set of suffixes to $j_1, \dots, j_t$. Thus Y is a direct sum of H-irreducible subgroups and we need therefore only consider the case in which Y itself is H-irreducible. Let $\overline{W}_j = W_1 \oplus \dots \oplus W_j$ and let j be the first integer $\leq n$ such that $Y \leq \overline{W}_j$. Then $\overline{W}_{j-1} \cap Y = 0$ by the irreducibility of Y and $\overline{W}_j = \overline{W}_{j-1} \oplus Y$ by the irreducibility of W_j. So Y is H-isomorphic with $W_j = X_{is}$ say. By the same argument applied with the W_j's rearranged so that all those H-isomorphic with Y come first (viz. $X_{i1}, X_{i2}, \dots$), we obtain $Y \leq V_i$, as required.

If $\xi \in G$, then as we have seen $X_{i1}\xi$ is H-irreducible. If $X_{i1}\xi$ is H-isomorphic with X_{j1}, then $X_{i1}\xi \leq V_j$ by (iii). If $u \to u'$ is an H-isomorphism of X_{i1} onto X_{is}, $u \in X_{i1}$, then $u\xi \to u'\xi$ is an H-isomorphism of $X_{i1}\xi$ onto $X_{is}\xi$, since $u\xi\eta = u(\xi\eta\xi^{-1})\xi \to u'(\xi\eta\xi^{-1})\xi = u'\xi\eta$ for $\eta \in H$, owing to $\xi\eta\xi^{-1} \in H$. Hence $X_{is}\xi \leq V_j$ for each $s = 1, 2, \dots$ and so $V_i\xi \leq V_j$. Here $j = j(i, \xi)$ depends only on i and on the automorphism $t_H(\xi)$ induced in H by transforming with ξ. It now follows that $V_i\xi = V_{j(i,\xi)}$ and the mapping $V_i \to V_i\xi$ $(i = 1, \dots, r)$ is a permutation of $V_1, \dots, V_r$. The representation of G by these permutations must be transitive, since otherwise V would not be G-irreducible. Hence each V_i is the direct sum of the same number ρ of H-irreducible subspaces and 8.9 is completely proved.

We call $V_1, \dots, V_r$ the <u>Wedderburn components</u> of V with respect to H.