(and replace 16 by 15 throughout)

§~~16~~ 15 Closure Properties of Classes of Groups. Groups of p-length 1.

(A) As in §8 (H) and 8.92, if $\mathfrak{X}$ is any class of groups, it is to be understood that (i) all unit groups belong to $\mathfrak{X}$ and (ii) if $G \in \mathfrak{X}$, and if $G \cong G_1$, then $G_1 \in \mathfrak{X}$.

In considering a given class $\mathfrak{X}$, it is usually desirable as a matter of routine to examine the closure properties of $\mathfrak{X}$. The most useful of these closure properties will be denoted by the small capitals

D, E, N, P, Q, R, S ;

and, if x is one of these, the following propositions state that $\mathfrak{X}$ has x or, as we shall also say, that $\mathfrak{X}$ is x-closed.

D. The direct product of two $\mathfrak{X}$-groups is an $\mathfrak{X}$-group.

E. Extensions of $\mathfrak{X}$-groups by $\mathfrak{X}$-groups are $\mathfrak{X}$-groups i.e. $K \triangleleft G$, $K \in \mathfrak{X}$ and $G/K \in \mathfrak{X}$ together imply $G \in \mathfrak{X}$.

N. Normal subgroups of $\mathfrak{X}$-groups are $\mathfrak{X}$-groups.

P. The product of two normal $\mathfrak{X}$-subgroups of any group is an $\mathfrak{X}$-group.

Q. Quotient groups of $\mathfrak{X}$-groups are $\mathfrak{X}$-groups.

R. A residual product of two $\mathfrak{X}$-groups is an $\mathfrak{X}$-group i.e. $K_i \triangleleft G$, $G/K_i \in \mathfrak{X}$ $(i=1,2)$ implies that $G/K_1 \cap K_2 \in \mathfrak{X}$.

S. Subgroups of $\mathfrak{X}$-groups are $\mathfrak{X}$-groups.

It is clear that, if x is any one of these closure properties, then the intersection of any number of x-closed classes of groups is also x-~~closure~~ closed. Hence, for an arbitrary class $\mathfrak{X}$, there is a uniquely defined smallest x-closed class containing $\mathfrak{X}$. This is called the x-closure of $\mathfrak{X}$ and will be denoted by $\mathrm{x}\mathfrak{X}$. The equation $\mathfrak{X} = \mathrm{x}\mathfrak{X}$ simply states that $\mathfrak{X}$ is x-closed. The following facts are clear.

Lemma 16·11 Let $\mathfrak{X}$ be any class of groups and let G be any group. Then (i) $G \in \mathrm{D}\mathfrak{X} \Leftrightarrow G$ is a direct product of $\mathfrak{X}$-subgroups.

(ii) $G \in \mathrm{E}\mathfrak{X} \Leftrightarrow G$ has a series whose factors are all $\mathfrak{X}$-groups.

(iii) $G \in \mathrm{N}\mathfrak{X} \Leftrightarrow G$ can be ~~normally~~ subnormally embedded in an $\mathfrak{X}$-group.

(iv) $G \in \mathrm{P}\mathfrak{X} \Leftrightarrow G$ is generated by its subnormal $\mathfrak{X}$-subgroups.

(v) $G \in Q\mathfrak{X} \Leftrightarrow$ G is a homomorphic image of an $\mathfrak{X}$-group

(vi) $G \in R\mathfrak{X} \Leftrightarrow$ G is a residual product of $\mathfrak{X}$-groups i.e. the normal subgroups K of G such that $G/K \in \mathfrak{X}$ intersect in 1.

(vii) $G \in S\mathfrak{X} \Leftrightarrow$ G can be embedded in an $\mathfrak{X}$-group.

(viii) $G \in QS\mathfrak{X} \Leftrightarrow$ G is isomorphic with a section of an $\mathfrak{X}$-group

(ix) $G \in RS\mathfrak{X} \Leftrightarrow$ G can be embedded in the ~~direct~~ Cartesian product of (a finite number of) $\mathfrak{X}$-groups

(x) $G \in QRS\mathfrak{X} \Leftrightarrow$ G is isomorphic with a section of a Cartesian product of $\mathfrak{X}$-groups.

(xi) $G \in EQN\mathfrak{X} \Leftrightarrow$ Every composition factor of G is a composition factor of some $\mathfrak{X}$-group.

In the last four cases, $XY\mathfrak{X}$ means $X(Y\mathfrak{X})$ of course. This class is X-closed by definition, but need not be Y-closed: in general the mapping $\mathfrak{X} \to XY\mathfrak{X}$ is not a closure operation i.e. XY, considered as an operator on classes of groups, will not be idempotent. The smallest class which is both X- and Y-closed and contains $\mathfrak{X}$ is in any case $\bigcup_{n=1}^{\infty}(XY)^n\mathfrak{X}$. But there are simplifications in special cases.

For example, (viii) shows that QS is idempotent and so $QS\mathfrak{X}$ is both Q- and S-closed. If we write $X \leq Y$ to mean that $X\mathfrak{X} \leq Y\mathfrak{X}$ for all classes $\mathfrak{X}$, then we have $SQ \leq QS$. Hence $SQS \leq$ ~~[illegible]~~ $QS^2 =$ Similarly $SR \leq RS$, so that RS is also a closure operator. More important, so is

$$V = QRS.$$

If G is any group, we define

$$\mathfrak{V}(G)$$

to consist of all groups H with the following property: if

$$f(\xi_1, \cdots, \xi_n) \equiv \xi_{i_1}^{\varepsilon_1}\xi_{i_2}^{\varepsilon_2}\cdots\xi_{i_n}^{\varepsilon_n} \qquad (\varepsilon\text{'s} = \pm 1) \qquad (1)$$

is any <u>word</u> (or <u>formal group-element</u>) in the indeterminates $\xi_1, \cdots, \xi_n$, such that $f(\xi_1, \cdots, \xi_n) = 1$ for all choices of the ξ's in G, then $f(\xi_1, \cdots, \xi_n) = 1$ for all choices of the ξ's in H. In other words, $H \in \mathfrak{V}(G)$ if and only if every law (or identical relation) holding in G also holds in H.

<u>Lemma 16.12</u> (i) Let G be any group, and let (G) be the class consisting of all groups of order 1 together with all groups isomorphic with Then $v(G) = \mathcal{V}(G)$.

(ii) A class $\mathfrak{X}$ of groups is v-closed if and only if $\mathcal{V}(G_1 \times G_2) \leq \mathfrak{X}$ for all G_1, G_2 in $\mathfrak{X}$.

Proof: (i) Obviously $\mathcal{V}(G)$ is Q-, R- and S-closed, hence v-closed. Hence $v(G) \leq \mathcal{V}(G)$. Suppose conversely that $H = \{\eta_1, \cdots, \eta_n\} \in \mathcal{V}(G)$. Consider the set F of all functions f of the form (1), where the arguments $\xi_1, \cdots, \xi_n$ range over G. Two such functions f_1 and f_2 are the same if and only if $f_1(\xi_1, \cdots, \xi_n) = f_2(\xi_1, \cdots, \xi_n)$ for all choices of the ξ's in G. Hence $|F| \leq g^{g^n}$, where $g = |G|$. Moreover, F is a group, if we define the multiplication and inversion of functions in the obvious way: in fact, it is a subgroup of a certain Cartesian power of G, the factors being in one-to-one correspondence with the ordered ~~multiplets~~ n-tuplets of elements of G. Hence $F \in SD(G) \leq RS(G)$. Since $H = \{\eta_1, \cdots, \eta_n\}$, every element of H has the form $f(\eta_1, \cdots, \eta_n)$ for some word f. Since $H \in \mathcal{V}(G)$, two such words f_1 and f_2 give the same value of H whenever $f_1(\xi_1, \cdots, \xi_n) = f_2(\xi_1, \cdots, \xi_n)$ for all choices of the ξ's in G. Hence we have a homomorphism of $\overset{F}{\cancel{H}}$ onto $\overset{H}{\cancel{F}}$ and so $H \in QRS(G) = v(G)$. Thus $\mathcal{V}(G) \leq v(G)$ and (i) is proved.

(ii) If $\mathfrak{X} = v\mathfrak{X}$, then $\mathcal{V}(G_1 \times G_2) \leq \mathfrak{X}$ for all G_1, G_2 in $\mathfrak{X}$ by (i). Conversely, suppose this condition is fulfilled. By 16.11(x), every G in $v\mathfrak{X}$ is isomorphic with a section of some group $H = G_1 \times \cdots \times G_n$ with $G_i \in \mathfrak{X}$, $i = 1, \cdots, n$. Hence $G \in v(H) = \mathcal{V}(H)$ and by hypothesis $\mathcal{V}(H) \leq \mathfrak{X}$. Hence $G \in \mathfrak{X}$ and so $v\mathfrak{X} = \mathfrak{X}$.

(B) Let $f_1, f_2, \cdots$ be any set of words. The class of all (finite) groups G in which the relations $f_1 = f_2 = \cdots = 1$ hold identically will be called a <u>variety</u> of groups. Thus $\mathcal{V}(G)$ is the smallest variety containing G. Every variety is v-closed, but the converse is not true. We may define

$\mathfrak{V}(\mathfrak{X})$ for any class $\mathfrak{X}$ to be the smallest variety containing $\mathfrak{X}$. Then $G \in \mathfrak{V}(\mathfrak{X})$ if and only if every law which holds in all $\mathfrak{X}$-groups also holds in G.

<u>Theorem 16·2</u> $\mathfrak{V}(\mathfrak{O}_p)$ contains all (finite) groups.

This seems to be due to Iwasawa. We prove it by showing that if f is given by (1), where $n > 0$ and $\varepsilon_{\alpha+1} = \varepsilon_\alpha$ whenever $i_{\alpha+1} = i_\alpha$, i.e. whenever f is a <u>non-trivial reduced word</u>, then there exists a finite p-group in which the law $f = 1$ does not hold. We may rewrite f in the form

$$f \equiv \xi_{j_1}^{m_1} \xi_{j_2}^{m_2} \cdots \xi_{j_r}^{m_r} \qquad (r > 0)$$

where $j_{\alpha+1} \neq j_\alpha$ and $m_1, m_2, \dots, m_r$ are integers $\neq 0$. Let

$$m_\alpha = p^{k_\alpha} q_\alpha, \quad k_\alpha \geqslant 0, \quad (p, q_\alpha) = 1.$$

Let $k = \sum_{\alpha=1}^{r} p^{k_\alpha}$ and let R be an additive elementary Abelian p-group with a basis consisting of all the formal products

$$v = u_{i_1} u_{i_2} \cdots u_{i_s} \qquad (0 \leq s \leq k),$$

including as the case $s = 0$ the empty product 1, where the suffixes i_α range from 1 to $j = \max_{\beta = 1, \dots, r} j_\beta$. If $v' = u_{l_1} u_{l_2} \cdots u_{l_t}$ we define $vv' = 0$ if $s + t > k$ and otherwise $vv' = u_{i_1} u_{i_2} \cdots u_{i_s} u_{l_1} u_{l_2} \cdots u_{l_t}$. ~~with~~ Extending this multiplication ~~from~~ of the basis elements of R by the distributive law to arbitrary elements, R becomes a ring. The elements of R of the form $1 + \sum_{v \neq 1} \lambda_v v$, with integers λ_v, form a multiplicative group G since we have, for any $w = \sum_{v \neq 1} \lambda_v v$, $w^{k+1} = 0$ and so

$$(1 + w)(1 - w + w^2 - \cdots + (-1)^k w^k) = 1.$$

G is a group of order p^g where $g = j + j^2 + \cdots + j^k$, and contains the elements $\eta_\alpha = 1 + u_\alpha$ $(\alpha = 1, 2, \dots, j)$. Let $\zeta = \eta_{j_1}^{m_1} \cdots \eta_{j_r}^{m_r}$. The coefficient of $u_{j_1}^{p^{k_1}} u_{j_2}^{p^{k_2}} \cdots u_{j_r}^{p^{k_r}}$ in ζ is then precisely

$$\binom{m_1}{p^{k_1}} \binom{m_2}{p^{k_2}} \cdots \binom{m_r}{p^{k_r}}$$

which is not divisible by p. Hence $\zeta \neq 1$ and the law $f = 1$ does not hold in
Thus 16·2 is proved.

Note that $\mathfrak{X} \to \mathfrak{V}(\mathfrak{X})$ is a closure operation. But it lacks the finitary character of the primary operations D, E, ... or V.

(C) <u>Lemma 16·31</u> (i) If $\mathfrak{X} = P\mathfrak{X}$, then every group G has a uniquely determined maximal normal $\mathfrak{X}$-subgroup $\mathfrak{X}G$. If in addition $\mathfrak{X} = N\mathfrak{X}$, then $\mathfrak{X}H = H \cap \mathfrak{X}G$ for every subnormal subgroup H of G

(ii) If $\mathfrak{X} = R\mathfrak{X}$, then every group G has a uniquely determined normal subgroup $G^{\mathfrak{X}}$ such that $G/G^{\mathfrak{X}} \in \mathfrak{X}$ and that $G/K \in \mathfrak{X}$ implies $G^{\mathfrak{X}} \leq K$. If in addition $\mathfrak{X} = Q\mathfrak{X}$, then $(G/K)^{\mathfrak{X}} = KG^{\mathfrak{X}}/K$ for every $K \triangleleft G$.

(iii) If $\mathfrak{X} = Q\mathfrak{X} = E\mathfrak{X}$, then $\mathfrak{X} = P\mathfrak{X}$ and every $\mathfrak{X}$-subgroup of G is contained in $\mathfrak{X}G$.

(iv) If $\mathfrak{X} = P\mathfrak{X} = E\mathfrak{X}$, then $\mathfrak{X}(G \bmod \mathfrak{X}G) = \mathfrak{X}G$.

(v) If $\mathfrak{X} = R\mathfrak{X} = E\mathfrak{X}$, then $(G^{\mathfrak{X}})^{\mathfrak{X}} = G^{\mathfrak{X}}$.

This is clear.

More interesting is

<u>Theorem 16·32</u> Suppose that $\mathfrak{X} = P\mathfrak{X} = N\mathfrak{X}$ and that $\mathfrak{X}$ contains some p-group $G \neq 1$. Then $\mathfrak{O}_p \leq \mathfrak{X}$ i.e. every p-group belongs to $\mathfrak{X}$.

Proof: G contains a subgroup C of order p and C sbn G. Hence $C \in \mathfrak{X}$ since $\mathfrak{X} = N\mathfrak{X}$. But $\mathfrak{X} = P\mathfrak{X}$ and so every elementary p-group belongs to $\mathfrak{X}$. Now let H be any p-group and let $K = C \wr H$. Then the base group $\overline{C}$ of K is an elementary p-group with a basis u_α $(\alpha \in H)$ such that $\beta^{-1}u_\alpha\beta = u_{\alpha\beta}$ for all $\alpha, \beta \in H$. Thus H is represented faithfully by automorphisms of $\overline{C}$. Let $A = \mathrm{Aut}\,\overline{C}$ and let S be a Sylow p-subgroup of A. Then $H \cong H_1$ where H_1 is some subgroup of S. Since H_1 sbn S and $\mathfrak{X} = N\mathfrak{X}$, it will be sufficient to show that $S \in \mathfrak{X}$; for then it will follow that $H \in \mathfrak{X}$.

Let $|H| = n$ and let $u_1, \dots, u_n$ be any basis of $\overline{C}$. Let $C_i = \{u_{i+1}, \dots, u_n\}$ so that $\overline{C} = C_0 > C_1 > \dots > C_n = 0$, where $\overline{C}$ is written additively. If α is any automorphism of $\overline{C}$ which centralizes each C_{i-1}/C_i, then $u_i\alpha = u_i +$ terms in $u_{i+1}, \dots, u_n = u_i + l_i(u_{i+1}, \dots, u_n)$. For any choice of the forms $l_1, \dots, l_n$, the elements $u_1 + l_1, \dots, u_n + l_n = u_n$ form a basis of $\overline{C}$; and so $u_i \to u_i + l_i$ $(i = 1, 2, \dots, n)$ defines an automorphism α of $\overline{C}$ of the kind described. Hence the group S of all such automorphis is of order $p^{\binom{n}{2}} = |A|_p$, and so S is a Sylow p-subgroup of A.

Let S_i be the group of all $\alpha \in S$ which centralize both C_i and $\overline{C}/C_i$. Clearly $S_i \lhd S$ and S_i is an elementary Abelian subgroup of order $p^{i(n-i)}$. Thus $S_i \in \mathfrak{X}$. Given $\alpha \in S$, define $\beta \in S_1$ by the condition $u_1\beta = u_1\alpha$. Then $\alpha\beta^{-1}$ leaves u_1 invariant and induces on C_1 an automorphism centralizing each C_{i-1}/C_i $(i = 2, 3, \ldots, n)$. By induction on n, we may assume that $\alpha\beta^{-1} \in S_n S_{n-1} \cdots S_2$. Then $\alpha \in S_n S_{n-1} \cdots S_1$. Hence $S = S_n S_{n-1} \cdots S_1$. Since each $S_i \lhd S$ and belongs to $\mathfrak{X} = P\mathfrak{X}$, it follows that $S \in P\mathfrak{X}$ as stated.

(D) By 16·11 (xi), any class $\mathfrak{Y}$ of the form $ERN\mathfrak{X}$ consists of all groups whose composition factors belong to a specified ~~class~~ set of simple groups. Every such class $\mathfrak{Y}$ is closed with respect to all seven primary operations D, E, ... with the possible exception of S. The most important classes of this kind are:

$\mathfrak{O}_\varpi$ = all ϖ-groups,

$\mathfrak{S}$ = all semisimple groups,

$\mathfrak{L}$ = all soluble groups.

In addition, the class of all ϖ-separable groups $\mathfrak{E}$ and the class of all ϖ-soluble groups are of this kind. And all these classes, with the exception of $\mathfrak{S}$, are S-closed.

The class

$\mathfrak{N}$ = all nilpotent groups

is closed with respect to all primary operators with the exception of E; while

$\mathfrak{A}$ = all Abelian groups

is a typical variety and is closed with respect to all the primary operators except P and E.

<u>Lemma 16·4?</u> Let ϖ be any set of primes, let $\mathfrak{X}$ be any class of groups and let $\mathfrak{Y}$ be the class of all ϖ-soluble groups whose S_ϖ-subgroups belong to $\mathfrak{X}$. Then any of the seven primary closure properties which belong to $\mathfrak{X}$ are inherited by $\mathfrak{Y}$

For example, suppose that $G = HK$ is the product of two normal $\mathfrak{Y}$-subgroups H and K. Then G is ϖ-soluble and if S is any S_ϖ-subgroup of G, then $S_1 = S \cap H$ and $S_2 = S \cap K$ are S_ϖ-subgroups of H and K respectively. Hence S_1 and S_2 belong to $\mathfrak{X}$. But $S = S_1 S_2$ and $S_i \lhd S$, $(i=1,2)$. Hence if $\mathfrak{X} = \mathrm{P}\mathfrak{X}$, we have $S \in \mathfrak{X}$ and so $G \in \mathfrak{Y}$. Thus $\mathfrak{Y} = \mathrm{P}\mathfrak{Y}$.

Exactly similar is the case $\mathfrak{X} = \mathrm{R}\mathfrak{X}$.

(E) If $\mathfrak{X}_1, \ldots, \mathfrak{X}_n$ are any classes of groups, we define $\mathfrak{X}_1 \mathfrak{X}_2 \cdots \mathfrak{X}_n$ to be the class of all groups G with a series

$$G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_n = 1$$

such that $G_{i-1}/G_i \in \mathfrak{X}_i$ for each i. This multiplication of classes is not in general associative. We have

$$(\mathfrak{X}_1 \mathfrak{X}_2)\mathfrak{X}_3 \leq \mathfrak{X}_1(\mathfrak{X}_2\mathfrak{X}_3) = \mathfrak{X}_1\mathfrak{X}_2\mathfrak{X}_3 .$$

<u>Lemma 16.51</u> If $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ are any three classes of groups, then either of the following two conditions is sufficient to ensure that $(\mathfrak{X}\mathfrak{Y})\mathfrak{Z} = \mathfrak{X}(\mathfrak{Y}\mathfrak{Z})$:

(i) $\mathfrak{Y} = \mathrm{Q}\mathfrak{Y}$ and $\mathfrak{Z} = \mathrm{P}\mathfrak{Z}$;

(ii) $\mathfrak{Y} = \mathrm{R}\mathfrak{Y}$ and $\mathfrak{Z} = \mathrm{N}\mathfrak{Z}$.

This is clear.

With regard to the inheritance of closure properties of $\mathfrak{X}$ and $\mathfrak{Y}$ by $\mathfrak{X}\mathfrak{Y}$, we have

<u>Lemma 16.52</u> (i) If $\mathfrak{X}$ and $\mathfrak{Y}$ are both closed with respect to any of $\mathrm{D}, \mathrm{S}, \mathrm{Q}, \mathrm{N}$, then so is $\mathfrak{X}\mathfrak{Y}$.

(ii) If $\mathfrak{X}$ and $\mathfrak{Y}$ are P-closed and $\mathfrak{X} = \mathrm{Q}\mathfrak{X}$, then $\mathfrak{X}\mathfrak{Y}$ is P-closed and $(\mathfrak{X}\mathfrak{Y})G = \mathfrak{X}(G \bmod \mathfrak{Y}G)$.

(iii) If $\mathfrak{X}$ and $\mathfrak{Y}$ are R-closed and $\mathfrak{Y} = \mathrm{N}\mathfrak{Y}$, then $\mathfrak{X}\mathfrak{Y}$ is R-closed and $G^{\mathfrak{X}\mathfrak{Y}} = (G^{\mathfrak{X}})^{\mathfrak{Y}}$.

For example, every p-soluble group G is contained in one of the classes $\mathfrak{O}_{p'}, \mathfrak{O}_{p'}\mathfrak{O}_p, \mathfrak{O}_{p'}\mathfrak{O}_p\mathfrak{O}_{p'}, \ldots \mathfrak{O}_p\mathfrak{O}_{p'} = \mathfrak{O}_{p'}(\mathfrak{O}_p\mathfrak{O}_{p'})^l$ for some integer l. The smallest such l is called the <u>p-length</u> of G and denoted by $l_p(G)$. By 16.52 (ii), the p-soluble groups of p-length at most l form a P-closed class, $\mathfrak{O}_{p'}\mathfrak{O}_p^{\,l}$ where

$\mathcal{P}_p = \mathcal{O}_p\mathcal{O}_{p'}$. If $\mathcal{P}_{p'} = \mathcal{O}_{p'}\mathcal{O}_p$, then $\mathcal{P}_{p'}^{l}$ is also a p-closed class, as is $\mathcal{P}_p^{l}$. Indeed the classes $\mathcal{P}_p^{l}$, $\mathcal{P}_{p'}^{l}$, $\mathcal{O}_{p'}\mathcal{P}_p^{l}$ and $\mathcal{O}_p\mathcal{P}_{p'}^{l}$ are closed with respect to all the primary operators except E.

If G is p-soluble of p-length l, then the series of radicals

$$1 \le \mathcal{O}_{p'}G < \mathcal{P}_pG < (\mathcal{O}_{p'}\mathcal{P}_p)G < \mathcal{P}_p^2G < \dots < \mathcal{P}_p^{l}G \le (\mathcal{O}_{p'}\mathcal{P}_p^{l})G \qquad (1)$$

is called the <u>upper p-series</u> of G; and the series

$$G \ge G^{\mathcal{O}_{p'}} \ge G^{\mathcal{P}_{p'}} > G^{\mathcal{P}_{p'}\mathcal{O}_{p'}} > G^{\mathcal{P}_{p'}^2} > \dots > G^{\mathcal{P}_{p'}^{l}} \ge G^{\mathcal{P}_{p'}^{l}\mathcal{O}_{p'}} = 1 \qquad (2)$$

is called the <u>lower p-series</u> of G. Upper and lower p-series of a p-soluble group are related rather like upper and lower central series of a nilpotent group. The k-th term from the right in (1) contains the k-th but not the (k+1)-th term from the right in (2).

Note that $\mathcal{P}_p$ is the class of all groups with a normal $S_{p'}$-subgroup.

(F) <u>Lemma 16.61</u> Let G be a p-soluble group.
(i) If $H = \mathcal{P}_pG$ and $K = \varphi(H).\mathcal{O}_{p'}G$, then $C_G(H/K) \le H$.
(ii) The intersection of the centralizers of all chief p-factors of G is H.

Proof: (i) Let $M = \mathcal{O}_{p'}G$. Then H/M is a p-group and K/M is its Frattini subgroup. Clearly, we may assume that $M = 1$. Let $C = C_G(H/K)$ and suppose if possible that $C \nleq H$. Then there is a chief factor L/H of G such that $L \le C$. (Note that $H \le C$ since H/K is Abelian). Since $H = \mathcal{P}_pG$, L/H is a p'-group and L has an $S_{p'}$-subgroup S such that $L = SH$. Since $L \le C$, we have $[H, S] \le K$ and so $[H, S] = 1$ by 13.2 (iii), since by hypothesis $M = 1$ and so H is a p-group. Hence L is the direct product of S and H, S char L, $S \lhd G$, $S \le M = \mathcal{O}_{p'}G$, contrary to $M = 1$. We conclude that $H = C$.

(ii) If N is the intersection of the centralizers of the chief p-factors of G, then every chief p-factor of N is a central factor of N. Let $N_1 = \mathcal{P}_pN$, $M_1 = \mathcal{O}_{p'}N$, $K_1 = \varphi(N_1).M_1$. By (i), if $N_1 < N$, there is a p'-element $\xi \in N$ which does not centralize N_1/K_1. But N_1/K_1 is an elementary p-group. Hence there must be some chief p-factor L/M of N with $K_1 \le M \le L \le N_1$ which is not centralized by ξ. This is a contradiction. Hence $N = \mathcal{P}_pN$. Since $N \lhd G$

it follows that $N \leq H = P_p G$. But clearly H centralizes every chief p-factor of G. Hence $N = H$.

<u>Lemma 16.62</u> Every A-group and every metanilpotent group is of p-length ≤ 1 for all primes p.

Here, the class of metanilpotent groups is $\mathfrak{N}^2 = \mathfrak{N}\mathfrak{N}$.

Proof: Let G be an A-group. We may assume that $O_{p'}G = 1$ since $G/O_{p'}G$ is an A-group of the same p-length as G. Let $H = O_p G$. Then every Sylow p-subgroup S of G contains H. Since S is Abelian, $S \leq C_G(H) \leq C_G(H/\varphi(H))$. By 16.61 (i), this implies that $S \leq H$, since $H = P_p G$ owing to $O_{p'}G = 1$. Hence G/H is a p'-group and so $l_p(G) \leq 1$.

Let G be metanilpotent. Again we may assume that $O_{p'}G = 1$. Then $F = \mathfrak{N}G$ is a p-group. Since $G \in \mathfrak{N}^2$, G/F is nilpotent and if S/F is its Sylow p-subgroup, then $S \triangleleft G$ and G/S is a p'-group. Thus $S = O_p G$ is a Sylow p-subgroup of G and $l_p(G) \leq 1$.

The class $\mathfrak{L}_1$ of soluble groups which are of p-length ≤ 1 for all p thus contains many of the more interesting special kinds of soluble group e.g. supersoluble groups, Z-groups, complemented groups, etc.

<u>Theorem 16.7</u> Let G be a p-soluble group. Then the following conditions are equivalent.

(i) $l_p(G) \leq 1$.

(ii) Every p-subgroup P of G is a Sylow p-subgroup of some subnormal subgroup of G i.e. $|P^{\cdots G} : P|$ is prime to p.

(iii) Every pronormal p-subgroup of G is a Sylow p-subgroup of some normal subgroup of G.

(iv) The automizer in G of every chief p-factor of G is a p'-group.

Proof: (i) $\Rightarrow$ (ii). For let S be a Sylow p-subgroup of G containing P. Since $l_p(G) \leq 1$, there exist normal subgroups L and K of G such that $L = SK$ and $S \cap K = 1$. Hence $L/K \cong S$; and, since P sbn S, we have KP sbn $KS = L$ and so KP sbn G. But K is a p'-group and hence P is a Sylow p-subgroup of KP.

(ii) $\Rightarrow$ (iii). Suppose P is a pronormal p-subgroup of G. By hypothesis, $|P^{\cdots G} : P|$ is prime to p. But by 6.65, the subnormal closure $P^{\cdots G}$ of P in G coincides with its normal closure $K = \{P^G\}$. Hence P is a Sylow p-subgroup of the normal subgroup K of G.

(iii) $\Rightarrow$ (i). Here we need the

<u>Lemma 16.71</u>. Let G be a p-soluble group and suppose that $l_p(G/K) < l = l_p(G)$ for all normal subgroups $K \neq 1$ of G. Then G is monolithic; its unique minimal normal subgroup M is an elementary Abelian p-group and $G = MS$ where S is a subgroup complementary to M in G, $M \cap S = 1$.

Proof. Suppose G had two different minimal normal subgroups M and M_1. By hypothesis, G/M and G/M_1 are of p-length at most $l-1$. But $M \cap M_1 = 1$ and the class $\mathcal{O}_{p'}\mathcal{O}_p^{\,l-1}$ is R-closed. Hence $l_p(G) \leq l-1$, a contradiction. Hence G is monolithic.

Since G is p-soluble, M is either an elementary Abelian p-group or else a p'-group. The second case is excluded since it would make $l_p(G/M) = l_p(G)$. Let $L = \mathcal{O}_{p'}(G \bmod M)$. Then $L > M$ since otherwise we should again have $l_p(G/M) = l_p(G)$. Hence $L = MT$, where T is an $S_{p'}$-subgroup of L. Let $S = N_G(T)$. Then $MS = G$ by the intravariance of T in L. Hence $M \cap S \lhd MS = G$, since M is Abelian. If $M \leq S$, then L would be the direct product of M and T and we should have T char L, $T \lhd G$, $T \neq 1$, contrary to the monolithic character of G. Hence $M \nleq S$ and so $M \cap S = 1$, by the minimality of M as a normal subgroup of G. This proves 16.71.

Now let G satisfy (iii) of 16.7. Let $K \lhd G$ and choose K to be maximal subject to $l_p(G/K) = l_p(G)$. Then $\Gamma = G/K$ satisfies the hypothesis of 16.71. By 6.68, every pronormal p-subgroup of Γ has the form KP/K where P is some pronormal p-subgroup of G. By hypothesis, there is a normal subgroup L of G such that P is a Sylow p-subgroup of L. Then KP/K is a Sylow p-subgroup of LK/K and $LK/K \lhd \Gamma$. Thus Γ also satisfies (iii). Hence we may assume without loss of generality that $G = \Gamma = MS$, $M \cap S = 1$, where M is a minimal normal subgroup of G and $|M| = p^m$. Suppose if possible that p divides $|S|$ and let P be a Sylow p-subgroup of S. Since M is a p-group, we have $P_1 = [M, P] < M$. Now $C = C_S(M) \triangleleft MS = G$ and, since G is monolithic, it follows that $C = 1$. But $P \neq 1$, by hypothesis. Hence $P_1 \neq 1$. By 6.69, $H = PP_1$ is a pronormal p-subgroup of G. Since $P_1 \neq 1$, we have $M \leq \{H^G\}$. Since $H \cap M = P_1 < M$, it follows that H is not a Sylow p-subgroup of $\{H^G\}$. This contradicts (iii). We conclude that S is a p'-group and so $l_p(G) = 1$.

(i) $\Rightarrow$ (iv). For if $l_p(G) = 1$, there are normal subgroups L and K of G such that $L = KS$, $K \cap S = 1$ where S is a Sylow p-subgroup of G. Then every chief p-factor of G is incident with one of the form E/D where $K \leq D < E \leq L$. By 9.3 (i), $C_G(E/D)$ contains L. But G/L is a p'-group and so $A_G(E/D)$ is a p'-group.

(iv) $\Rightarrow$ (i). Here, as in the proof of (iii) $\Rightarrow$ (i), we replace G by a quotient group of smallest order having the same p-length as G. Then denoting this quotient group still by G, we have $G = MS$, $M \cap S = 1$, where M is the unique minimal normal subgroup of G and so $C_S(M) = 1$. It follows that $S \cong A_G(M)$. Hence S is a p'-group and $l_p(G) = 1$.

Remark: The equivalence of 16.7 (i) and (ii) is proved in a rather more general form (not restricted by the hypothesis that G is p-soluble) by Wielandt.

(G) <u>Lemma 16.8</u> (i) The class of all groups G with $O_{\varpi}G = 1$ is D-, E-, R- and N- closed, but not P-, Q- or S- closed

(ii) Dually, the class of all groups G with $G = G^{O_{\varpi}}$ (i.e. the groups which are generated by their ϖ'-elements) is D-, E-, P- and Q- closed, but not N-, R- or S- closed.